November, 2000
OU-HET 368,
hep-th/0011028

Note on Open String/D-brane System and Noncommutative Soliton *

H. Itoyama†

Department of Physics, Graduate School of Science, Osaka University, Toyonaka, Osaka 560-0043, Japan

Abstract

This is a summary of a series of papers [1, 2, 3] written with B. Chen, T. Matsuo and K. Murakami on a $p-p', (p < p')$ open string with B_{ij} field, which has led us to the explicit identification of the Dp-brane with the noncommutative projector soliton via the gaussian damping factor. A lecture given at Summer Institute 2000, FujiYoshida, Yamanashi, Japan, at August 7-14, 2000.

*This work is supported in part by the Grant-in-Aid for Scientific Research (12640272, 12014210) from the Ministry of Education, Science and Culture, Japan

†e-mail address: itoyama@funpth.phys.sci.osaka-u.ac.jp

I. Introduction

The $p - p'$, $(p < p')$ open string is one of our favorite systems to discuss gauge fields and topological defects appearing on the worldvolume of the bigger brane. The presence of B field makes this situation more interesting and, in some sense, more tractable. (See, for example, [4, 5].) A little less than a year ago, we have started to look at this system in string perturbation theory. I would like to present several interesting results coming out from our computation and understading of the phenomena.

There are two distinct points of view to the system.

1. the view from the smaller (Dp) brane : this may be called moduli space point of view.
2. the view from the bigger (Dp') brane : this may be called spacetime point of view, which we have put forward in our investigation.

In what follows, the Dp-brane extends in the $(x^0, x^1, \ldots, x^p)$-directions and the Dp'-brane extends in the $(x^0, x^1, \ldots, x^{p'})$-directions with the D$p$-brane inside. The D$p$-brane worldvolume contains the boundary $\sigma = 0$ while the Dp'-brane worldvolume contains the boundary $\sigma = \pi$.

II. Description of the System and Worldsheet Properties

1. the action of the NSR superstring in the constant B background:

$$S = \frac{1}{2\pi} \int d^2\xi \int d\theta d\overline{\theta} \left(g_{\mu\nu} + 2\pi\alpha' B_{\mu\nu} \right) \overline{D}\mathbf{X}^\mu(\mathbf{z}, \overline{\mathbf{z}}) D\mathbf{X}^\nu(\mathbf{z}, \overline{\mathbf{z}}) \ , \tag{2.1}$$

where $\mathbf{z} = (z, \theta)$ and $\overline{\mathbf{z}} = (\overline{z}, \overline{\theta})$, $D = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial z}$ and $\overline{D} = \frac{\partial}{\partial\overline{\theta}} + \overline{\theta}\frac{\partial}{\partial\overline{z}}$, $z = \xi^1 + i\xi^2$ and $\overline{z} = \xi^1 - i\xi^2$ and $z = e^{\tau + i\sigma}$ and $\overline{z} = e^{\tau - i\sigma}$.

$$g_{\mu\nu} = \left(\begin{array}{c|c|c} -1 & & \\ \hline & g_{ij} & \\ \hline & & \begin{array}{ccc} 1 & & \\ & \ddots & \\ & & 1 \end{array} \end{array} \right) \ , \qquad g_{ij} = \varepsilon\delta_{ij} \quad (i, j = 1, \ldots, p') \ , \tag{2.2}$$

$$B_{ij} = \frac{\varepsilon}{2\pi\alpha'} \begin{pmatrix} 0 & b_1 & & & \\ -b_1 & 0 & & & \\ & & 0 & b_2 & \\ & & -b_2 & 0 & \\ & & & & \ddots \end{pmatrix} \quad (i,j=1,\ldots,p')\ , \quad \text{otherwise } B_{\mu\nu}=0\ . \quad (2.3)$$

2. the boundary conditions for the string coordinates in the NS sector:

$$\begin{aligned}
&D\mathbf{X}^0 - \overline{D}\mathbf{X}^0\Big|_{\sigma=0,\pi\ \ \theta=\overline{\theta}} = 0\ , \quad D\mathbf{X}^{p'+1,\ldots,9} + \overline{D}\mathbf{X}^{p'+1,\ldots,9}\Big|_{\sigma=0,\pi\ \ \theta=\overline{\theta}} = 0\ ,\\
&g_{kl}(D\mathbf{X}^l - \overline{D}\mathbf{X}^l) + 2\pi\alpha' B_{kl}(D\mathbf{X}^l + \overline{D}\mathbf{X}^l)\Big|_{\sigma=0,\pi\ \ \theta=\overline{\theta}} = 0 \quad (k,l=1,\ldots,p)\\
&D\mathbf{X}^i + \overline{D}\mathbf{X}^i\Big|_{\sigma=0\ \ \theta=\overline{\theta}} = g_{ij}(D\mathbf{X}^j - \overline{D}\mathbf{X}^j) + 2\pi\alpha' B_{ij}(D\mathbf{X}^j + \overline{D}\mathbf{X}^j)\Big|_{\sigma=\pi\ \ \theta=\overline{\theta}} = 0\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (i,j=p+1,\ldots,p')\ . \quad (2.4)
\end{aligned}$$

One may say that the scale of noncommutativity is contained in these boundary conditionss.

3. quantization and the mode expansion

We concentrate only on the x^i-directions ($i = p+1,\ldots,p'$). We complexify the string coordinates $\mathbf{X}^i(\mathbf{z},\overline{\mathbf{z}})$ in these directions as

$$\begin{aligned}
\mathbf{Z}^I(\mathbf{z},\overline{\mathbf{z}}) &= \mathbf{X}^{2I-1}(\mathbf{z},\overline{\mathbf{z}}) + i\mathbf{X}^{2I}(\mathbf{z},\overline{\mathbf{z}}) = \sqrt{\frac{2}{\alpha'}}Z^I(z,\bar{z}) + i\theta\Psi^I(z) + i\overline{\theta}\widetilde{\Psi}^I(\overline{z})\ ,\\
\overline{\mathbf{Z}}^{\overline{I}}(\mathbf{z},\overline{\mathbf{z}}) &= \mathbf{X}^{2I-1}(\mathbf{z},\overline{\mathbf{z}}) - i\mathbf{X}^{2I}(\mathbf{z},\overline{\mathbf{z}}) = \sqrt{\frac{2}{\alpha'}}\overline{Z}^{\overline{I}}(z,\bar{z}) + i\theta\overline{\Psi}^{\overline{I}}(z) + i\overline{\theta}\overline{\widetilde{\Psi}}^{\overline{I}}(\overline{z})\ , \qquad (2.5)
\end{aligned}$$

where $I,\overline{I} = \frac{p+2}{2},\ldots,\frac{p'}{2}$.

$$\begin{aligned}
Z^I(z,\overline{z}) &= i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbf{Z}}\frac{\alpha^I_{n-\nu_I}}{n-\nu_I}\left(z^{-(n-\nu_I)} - \overline{z}^{-(n-\nu_I)}\right)\ ,\\
\overline{Z}^{\overline{I}}(z,\overline{z}) &= i\sqrt{\frac{\alpha'}{2}}\sum_{m\in\mathbf{Z}}\frac{\overline{\alpha}^{\overline{I}}_{m+\nu_I}}{m+\nu_I}\left(z^{-(m+\nu_I)} - \overline{z}^{-(m+\nu_I)}\right)\ , \qquad (2.6)
\end{aligned}$$

where ν_I are defined by

$$e^{2\pi i\nu_I} = -\frac{1+ib_I}{1-ib_I}\ , \quad 0<\nu_I<1\ . \qquad (2.7)$$

Now we can introduce the open string metric G^{IJ}, $G^{\overline{I}\overline{J}}$, $G^{I\overline{J}}$ and $G^{\overline{I}J}$ concerning the $x^{p+1},\ldots,x^{p'}$ directions,

$$G^{IJ} = G^{\overline{I}\overline{J}} = 0\ , \quad G^{I\overline{J}} = G^{\overline{J}I} = \frac{2}{\varepsilon(1+b_I^2)}\delta^{I\overline{J}}\ . \qquad (2.8)$$

The commutation relations are

$$
\begin{aligned}
&[\alpha^I_{n-\nu_I}\,,\,\overline{\alpha}^{\overline{J}}_{m+\nu_J}] = \frac{2}{\varepsilon}\delta^{I\overline{J}}(n-\nu_I)\delta_{n+m}\ ,\\
&\{b^I_{r-\nu_I}\,,\,\overline{b}^{\overline{J}}_{s+\nu_J}\} = \frac{2}{\varepsilon}\,\delta^{I\overline{J}}\delta_{r+s}\ , \qquad \{d^I_{n-\nu_I}\,,\,\overline{d}^{\overline{J}}_{m+\nu_J}\} = \frac{2}{\varepsilon}\,\delta^{I\overline{J}}\delta_{n+m}\ .
\end{aligned}
\tag{2.9}
$$

4. the oscillator vacuum

The oscillator vacuum $|\sigma,s\rangle \equiv |\sigma\rangle \bigotimes |s\rangle$:

$$
|\sigma\rangle = \bigotimes_I |\sigma_I\rangle \qquad \text{with} \qquad \begin{cases} \alpha^I_{n-\nu_I}\,|\sigma_I\rangle = 0 & n>\nu_I \\ \overline{\alpha}^{\overline{I}}_{m+\nu_I}\,|\sigma_I\rangle = 0 & m>-\nu_I \end{cases}\ . \tag{2.10}
$$

$$
|s\rangle = \bigotimes_I |s_I\rangle \qquad \text{with} \qquad \begin{cases} b^I_{r-\nu_I}\,|s_I\rangle = 0 & r \geq \frac{1}{2} \\ \overline{b}^{\overline{I}}_{s+\nu_I}\,|s_I\rangle = 0 & s \geq \frac{1}{2} \end{cases}\ , \tag{2.11}
$$

A twist field $\sigma^+_I(\xi^1)$ and an anti-twist field $\sigma^-_I(\xi^1)$, both of which are mutually non-local with respect to Z^I and $\overline{Z}^I$, are located at the origin and at infinity on the plane respectively [6, 7]. They create a branch cut between themselves. The incoming vacuum $|\sigma_I\rangle$ defined in eq. (2.10) should be interpreted as being excited from the $SL(2,\mathbf{R})$-invariant vacuum $|0\rangle$ by the twist field σ^+_I:

$$
|\sigma_I\rangle = \lim_{\xi^1\to 0} \sigma^+_I(\xi^1)\,|0\rangle\ . \tag{2.12}
$$

A similar comment applies to the spin field.

5. spectrum

Reflecting the vacuum sea filling, the spectrum of the $p-p'$ system cannot be given generically. The spectrum of each individual case of $p-p'$ has been fully analyzed in [1]. (See also [5] for the $0-4$ case.) If we finetune the sign of b_I, a large number of light states appear in the limit. To be more precise, these light states are obtained by acting the several low-lying fermionic modes on the oscillator vacuum and multiplying by an arbitrary polynomial consisting of the lowest bosonic mode. This latter bosonic mode is the one which has failed to become a momentum due to the boundary condition of the $p-p'$ open string and is responsible for an infinite number of nearly degenerate low-lying states.

6. two-point function on superspace

Let

$$
\boldsymbol{\mathcal{G}}^{I\overline{J}}\,(\mathbf{z}_1,\overline{\mathbf{z}}_1|\mathbf{z}_2,\overline{\mathbf{z}}_2) \equiv \langle\sigma,s|\,\mathcal{R}\mathbf{Z}^I(\mathbf{z}_1,\overline{\mathbf{z}}_1)\overline{\mathbf{Z}}^{\overline{J}}(\mathbf{z}_2,\overline{\mathbf{z}}_2)\,|\sigma,s\rangle\ . \tag{2.13}
$$

When restricted onto the worldsheet boundary on the Dp'-brane worldvolume, this becomes

$$\boldsymbol{\mathcal{G}}^{I\overline{J}}\left(-e^{\tau_1},\theta_1|-e^{\tau_2},\theta_2\right)=4G^{I\overline{J}}\,\mathcal{H}\left(\nu_I;\frac{e^{\tau_1}}{e^{\tau_2}-\theta_1\theta_2}\right)+\epsilon(\tau_1-\tau_2)\frac{4}{\varepsilon}\frac{\delta^{I\overline{J}}}{1+b_I^2}\pi b_I\ , \tag{2.14}$$

where $\mathcal{H}(\nu;z)$ is defined by using the hypergeometric series as

$$\mathcal{H}(\nu;z)=\begin{cases}\mathcal{F}\left(1-\nu_I;\dfrac{1}{z}\right)-\dfrac{\pi}{2}b_I=\displaystyle\sum_{n=0}^{\infty}\frac{z^{-n-1+\nu_I}}{n+1-\nu_I}-\frac{\pi}{2}b_I & \text{for } |z|>1\\[2ex] \mathcal{F}\left(\nu_I;z\right)+\dfrac{\pi}{2}b_I=\displaystyle\sum_{n=0}^{\infty}\frac{z^{n+\nu_I}}{n+\nu_I}+\frac{\pi}{2}b_I & \text{for } |z|<1\ .\end{cases} \tag{2.15}$$

The function $\mathcal{F}(\nu\,;\,z)$ is defined as

$$\mathcal{F}(\nu\,;\,z)=\frac{z^{\nu}}{\nu}\,F(1,\nu;1+\nu;z)=\sum_{n=0}^{\infty}\frac{1}{n+\nu}z^{n+\nu}\ , \tag{2.16}$$

and $F(a,b;c;z)$ is the hypergeometric function. By using the hypergeometric series, we find that

$$\mathcal{F}(1-\nu_I;1)-\mathcal{F}(\nu_I;1)=-\sum_{n=-\infty}^{\infty}\frac{1}{n+\nu_I}=-\pi\cot\left(\pi\nu_I\right)=\pi b_I\ . \tag{2.17}$$

This means that, as is pointed out in [1], the noncommutativity on the D-brane worldvolume in the p-p' system is the same as that in the p-p system.

7. renormal ordering and subtracted two-point function

The contents of this subsection are crucial ingredients of the derivation of the string amplitudes presented in next section and are responsible for the major spacetime properties of the system. We have two types of vacuum: the one is the $SL(2,\mathbf{R})$-invariant vacuum and the other is the oscillator vacuum. We will use the symbols $:\ :$ and $\circ\!\!\circ\ \circ\!\!\circ$ to denote the normal orderings with respect to the $SL(2,\mathbf{R})$-invariant vacuum and the oscillator vacuum respectively. The $\circ\!\!\circ\ \circ\!\!\circ$-normal ordered product for the free fields in the x^i-directions ($i=p+1,\ldots,p'$) is

$$\circ\!\!\circ\,\mathbf{Z}^I(\mathbf{z}_1,\overline{\mathbf{z}}_1)\overline{\mathbf{Z}}^{\overline{J}}(\mathbf{z}_2,\overline{\mathbf{z}}_2)\,\circ\!\!\circ=\mathcal{R}\mathbf{Z}^I(\mathbf{z}_1,\overline{\mathbf{z}}_1)\overline{\mathbf{Z}}^{\overline{J}}(\mathbf{z}_2,\overline{\mathbf{z}}_2)-\boldsymbol{\mathcal{G}}^{I\overline{J}}(\mathbf{z}_1,\overline{\mathbf{z}}_1|\mathbf{z}_2,\overline{\mathbf{z}}_2)\ , \tag{2.18}$$

for $I,\overline{J}=\frac{p+2}{2},\ldots,\frac{p'}{2}$. Here $\boldsymbol{\mathcal{G}}^{I\overline{J}}(\mathbf{z}_1,\overline{\mathbf{z}}_1|\mathbf{z}_2,\overline{\mathbf{z}}_2)$ is the two-point function defined in eq. (2.13). The formula of the renormal ordering takes the form of

$$:\boldsymbol{\mathcal{O}}:=\exp\left(\int d^2\mathbf{z}_1d^2\mathbf{z}_2\,\boldsymbol{\mathcal{G}}_{\rm sub}{}^{I\overline{J}}(\mathbf{z}_1,\overline{\mathbf{z}}_1|\mathbf{z}_2,\overline{\mathbf{z}}_2)\frac{\delta}{\delta\mathbf{Z}^I(\mathbf{z}_1,\overline{\mathbf{z}}_1)}\frac{\delta}{\delta\overline{\mathbf{Z}}^{\overline{J}}(\mathbf{z}_2,\overline{\mathbf{z}}_2)}\right)\circ\!\!\circ\,\boldsymbol{\mathcal{O}}\,\circ\!\!\circ\ . \tag{2.19}$$

Here $\boldsymbol{\mathcal{G}}_{\rm sub}{}^{I\overline{J}}(\mathbf{z}_1,\overline{\mathbf{z}}_1|\mathbf{z}_2,\overline{\mathbf{z}}_2)$ is the subtracted two-point function defined as

$$\begin{aligned}\boldsymbol{\mathcal{G}}_{\rm sub}{}^{I\overline{J}}(\mathbf{z}_1,\overline{\mathbf{z}}_1|\mathbf{z}_2,\overline{\mathbf{z}}_2)&\equiv\langle\sigma,s|:\mathbf{Z}^I(\mathbf{z}_1,\overline{\mathbf{z}}_1)\overline{\mathbf{Z}}^{\overline{J}}(\mathbf{z}_2,\overline{\mathbf{z}}_2):|\sigma,s\rangle\\&=\boldsymbol{\mathcal{G}}^{I\overline{J}}(\mathbf{z}_1,\overline{\mathbf{z}}_1|\mathbf{z}_2,\overline{\mathbf{z}}_2)-\mathbf{G}^{I\overline{J}}(\mathbf{z}_1,\overline{\mathbf{z}}_1|\mathbf{z}_2,\overline{\mathbf{z}}_2)\ ,\end{aligned} \tag{2.20}$$

$\mathbf{G}^{I\overline{J}}(\mathbf{z}_1, \overline{\mathbf{z}}_1|\mathbf{z}_2, \overline{\mathbf{z}}_2)$ is the two-point function defined with respect to the $SL(2,\mathbf{R})$-invariant vacuum and take the well-known form [4, 5].

III. Scattering Amplitudes

Let us specify the process of our interest. At an initial state, we prepare a Dp-brane which is at rest and which lies in the worldvolume of a Dp'-brane. We place the tachyon (the lowest mode) of a $p-p'$ open string which carries a momentum $k_{1\mu}$, $\mu = 0\cdots p$ along the Dp-brane worldvolume. In addition, $N-2$ noncommutative $U(1)$ photons carrying momenta k_{aM} $a = 3\cdots N$, $M = 0\cdots p'$ in $p'+1$ dimensions are present. They get absorbed into the Dp-brane. At a final state, the Dp-brane is found to be present and the momentum of the tachyon is measured to be $-k_{2\mu}$ along the Dp-brane worldvolume. We will examine the tree scattering amplitude of this process both from string perturbation theory of the D-brane/open string system in the zero slope limit and from perturbation theory of the field theory action proposed in [2]. We will find that computations from both sides in fact agree by identifying the Dp-brane with an initial/final configuration representing a noncommutative soliton. Let

$$\kappa_I = \frac{1}{2}\left(k_{2I-1} - ik_{2I}\right)\ , \qquad \overline{\kappa}_{\overline{I}} = \frac{1}{2}\left(k_{2I-1} + ik_{2I}\right)\ ;$$

$$e_I(k) = \frac{1}{2}\Big(\zeta_{2I-1}(k) - i\zeta_{2I}(k)\Big)\ , \qquad \overline{e}_{\overline{I}}(k) = \frac{1}{2}\Big(\zeta_{2I-1}(k) + i\zeta_{2I}(k)\Big)\ . \tag{3.1}$$

Let (NC) denote

$$\begin{aligned}
(\mathrm{NC}) \quad &= \sum_{1\leq a<a'\leq N} \frac{i}{2}\epsilon(x_a - x_{a'}) \sum_{i,j=1}^{p} \theta^{ij} k_{ai} k_{a'j} \\
&\quad - \sum_{3\leq c<c'\leq N} \epsilon(x_c - x_{c'}) \sum_{I,\overline{J}} \alpha' \frac{2\delta^{I\overline{J}}\pi b_I}{\varepsilon(1+b_I^2)} \left(\kappa_{cI}\overline{\kappa}_{c'\overline{J}} - \overline{\kappa}_{c\overline{J}}\kappa_{c'I}\right) \\
&= \sum_{1\leq a<a'\leq N} \frac{i}{2}\epsilon(x_a - x_{a'}) \sum_{\mu,\lambda=0}^{p'} \theta^{\mu\lambda} k_{a\mu} k_{a'\lambda}\ ,
\end{aligned} \tag{3.2}$$

with $k_{1j} = k_{2j} = 0$ for $(j = p+1, \ldots, p')$. Space permits us only the final form of the N point amplitude:

$$\begin{aligned}
A_N = c(2\pi)^{p+1} \prod_{\mu=0}^{p} \delta\left(\sum_{e=1}^{N} k_{e\mu}\right) \int \prod_{a=4}^{N} dx_a \prod_{a'=3}^{N} d\theta_{a'} d\eta_{a'} \exp \mathcal{C}_{a'}(\nu_I) \\
\times \prod_{c=4}^{N} \left[x_c^{-\alpha' s_c + \alpha' m_T^2} (1-x_c)^{2\alpha' k_3 \overset{\cdot}{(p)} k_c} \right] \prod_{4\leq c<c'\leq N} (x_c - x_{c'})^{2\alpha' k_{c'} \overset{\cdot}{(p)} k_c}
\end{aligned}$$

$$\times \prod_{3\leq c<c'\leq N} \exp\left[-2\alpha' \sum_{I,\overline{J}} G^{I\overline{J}} \left\{\kappa_{cI}\overline{\kappa}_{c'\overline{J}}\mathcal{H}\left(\nu_I;\frac{x_c}{x_{c'}}\right)+\overline{\kappa}_{c\overline{J}}\kappa_{c'I}\mathcal{H}\left(\nu_I;\frac{x_{c'}}{x_c}\right)\right\}\right]$$

$$\times \exp\left(\mathrm{NC}\right) \exp\left([0,2]+[2,0]+[1,1]+[2,2]\right)\Big|_{x_1=0,x_2=\infty,x_3=1} . \tag{3.3}$$

This expression is regarded as an $SL(2,\mathbf{R})$ invariant integral (Koba-Nielsen) representation for the amplitude of our concern. Let us list several features which are distinct from the corresponding formula in the case of a p-p open string. (See [8]).

1. The term denoted by $\exp\left(\mathrm{NC}\right)$ which originated from the noncommutativity of the worldvolume extends into both the $x^1,\ldots,x^p$ directions and the remaining $x^{p+1},\ldots,x^{p'}$ directions.
2. To each external vector leg, we have a momentum dependent multiplicative factor $\exp\mathcal{C}(\nu_I)$.
3. A new tensor J has appeared.
4. There are parts in the expression which are expressible in terms of the momenta of the tachyons, the momenta and the polarization tensors of the vectors and J alone, using the inner product with respect to the open string metric. These parts come, however, with a host of other parts which do not permit such generic description in terms of the inner product.

The terms containing θ_a and η_a are classified by the number of η_a and by the number of θ_a, which we designate respectively by the first and by the second entry inside the bracket. These are given as $[0,2],[2,0],[1,1]$, and $[2,2]$. For $N=3$ case, we pick up θ_3 and η_3 from $[1,1]$ to saturate the Grassmann integrations. For $N=4$ case, we pick up terms from $[2,2]+[0,2][2,0]+\frac{1}{2}[1,1]^2$.

IV. The Zero Slope Limit and Noncommutative Soliton

We focus on the nontrivial zero slope limit of the amplitude.[1] The zero slope limit is defined as

$$\begin{aligned}\alpha' &\sim \varepsilon^{1/2}\rightarrow 0 ,\\ g &\sim \varepsilon\rightarrow 0 ,\\ |b_I| &\sim \varepsilon^{-1/2}\rightarrow \infty .\end{aligned} \tag{4.1}$$

[1] In this and the next sections, the spacetime index $M,N\cdots$ run from 0 to p', $\mu,\nu\cdots$ from 0 to p and $m,n\cdots$ from $p+1$ to p'.

This limit keeps $\alpha' b_I$ finite:

$$\alpha' b_I \rightarrow \beta_I \quad . \tag{4.2}$$

In terms of the open string metric and the noncommutativity parameter, this implies

$$\frac{1}{2\pi} \left(JG\theta\right)_{2I-1}^{\;\;2I-1} = \frac{1}{2\pi} \left(JG\theta\right)_{2I}^{\;\;2I} = \beta_I \quad . \tag{4.3}$$

In addition, the following limit is taken without loss of generality:

$$\nu \equiv \nu_{\frac{p+2}{2}} \rightarrow 1 \;, \quad \nu_{\tilde{I}} \rightarrow 0 \;, \;\; \text{for } \;\tilde{\mathrm{I}} \neq \frac{\mathrm{p}+2}{2} \quad , \tag{4.4}$$

so that

$$b_{\frac{p+2}{2}} \rightarrow +\infty \quad , \quad b_{\tilde{I}} \rightarrow -\infty \quad . \tag{4.5}$$

The exponential multiplicative factor $\exp \mathcal{C}(\{\nu_I\})$ becomes in the zero slope limit

$$\exp \mathcal{C}(\{\nu_I\}) \rightarrow \exp\left(-\pi \sum_{I,\bar{J}} |\beta_I|\, \kappa_I \overline{\kappa}_{\overline{J}} G^{I\overline{J}}\right) = \exp\left(-\frac{\pi}{2} \sum_I |\beta_I| \left(k \underset{(p,p')}{\odot} k\right)_I\right) \equiv D\left(k_m\right) \quad . \tag{4.6}$$

This factor is originally associated with each vector propagating into the $x^{p+1} \sim x^{p'}$ directions. We will refer to this as gaussian damping factor (g.d.f.). Replacing $c\sqrt{\frac{\alpha'}{2}}$ by the initial and final wave functions of the tachyon and the noncommutative $U(1)$ photon, which we should insert together with the vertex operators, we obtain

$$\lim \mathcal{A}_3 = (2\pi)^{p+1} \prod_{\mu=0}^{p} \delta\left(\sum_{a=1}^{3} k_{a\mu}\right) \frac{1}{(2\pi)^{p/2}} \frac{1}{\sqrt{2\omega_{\vec{k}_2}}} \frac{1}{(2\pi)^{p/2}} \frac{1}{\sqrt{2\omega_{\vec{k}_1}}} \frac{1}{(2\pi)^{p'/2}} \frac{1}{\sqrt{2|\vec{k}_3|}}$$
$$\times \left\{ (k_2 - k_1) \underset{(p)}{\cdot} \zeta_3 - i k_3 \underset{(p,p')}{\cdot} J \zeta_3 \right\} D(k_{3m}) e^{\frac{i}{2}\theta^{ij} k_{1i} k_{2j}} \quad . \tag{4.7}$$

Let us consider the following factor seen as an exponent of eq. (3.3):

$$P_{c'c} \equiv -2\alpha' \sum_{I,\overline{J}} G^{I\overline{J}} \left(\kappa_{cI} \overline{\kappa}_{c'\overline{J}} \mathcal{H}\left(\nu_I; \frac{x_c}{x_{c'}}\right) + \overline{\kappa}_{c\overline{J}} \kappa_{c'I} \mathcal{H}\left(\nu_I; \frac{x_{c'}}{x_c}\right) \right) \quad . \tag{4.8}$$

We see that, in any region contributing to the zero slope limit, this factor $P_{c'c}$ contains precisely the identical constant piece $-\pi \sum_I |\beta_I| (k_c \underset{(p,p')}{\odot} k_{c'})_I$ in the limit. Multiplying $\prod_{3\leq c<c'\leq N} \exp\left[-\pi \sum_I \mid \beta_I \mid\right.$ by $\prod_{a=3}^{N} D(k_{am})$, (see eq. (4.6)), we find that the amplitude A_N contains an overall multiplicative factor

$$D\left(\sum_{a=3}^{N} k_{am}\right) = \exp\left[-\frac{\pi}{2} \sum_I |\beta_I| \left((\sum_{a=3}^{N} k_a) \underset{(p,p')}{\odot} (\sum_{b=3}^{N} k_b) \right)_I \right] \quad , \tag{4.9}$$

which depends upon the total photon momentum alone. We have thus seen that the string amplitude in fact has resummed and lifted the approximate infinite degeneracy of the spectrum by evaluating its effect as an exponential factor and that this lifting has rendered the net g.d.f. of the amplitude to depend only upon the total momentum of the incoming photons.

We turn to the four point amplitude in the zero slope limit. After lifting the infinite degeneracy due to the lowest bosonic mode, we still have the contributions from several nearly degenerate states due to the lowest fermionic modes. See eq. (5.8) of [2] for the complete formula in the zero slope limit which contains the above contributions as well. Picking up only those parts of the amplitude in which the state with the lowest mass (tachyon) participates, we find

$$\begin{aligned}
\lim \mathcal{A}_4 \;=\; & -2i(2\pi)^{p+1}\prod_{\mu=0}^{p}\delta\left(\sum_{a=1}^{4}k_{a\mu}\right) D\left(k_{3m}+k_{4m}\right)\,\exp\left(\frac{i}{2}\theta^{\mu\nu}k_{1\mu}k_{2\nu}+\frac{i}{2}\theta^{MN}k_{3M}k_{4N}\right)\\
& \left[\frac{1}{t-m_T^2}\frac{1}{2}\left\{\left(k_2-(k_1+k_4)\right)\underset{(p)}{\cdot}\zeta_3 - ik_3\underset{(p,p')}{\cdot}J\zeta_3\right\}\right.\\
& \qquad\left\{\left((k_2+k_3)-k_1\right)\underset{(p)}{\cdot}\zeta_4 - ik_4\underset{(p,p')}{\cdot}J\zeta_4\right\}\\
& +\frac{1}{s}\left\{\left((k_2-k_1)\underset{(p)}{\cdot}\zeta_3 - i(k_3+k_4)\underset{(p,p')}{\cdot}J\zeta_3\right)k_3\underset{(p')}{\cdot}\zeta_4\right.\\
& \qquad-\left((k_2-k_1)\underset{(p)}{\cdot}\zeta_4 - i(k_3+k_4)\underset{(p,p')}{\cdot}J\zeta_4\right)k_4\underset{(p')}{\cdot}\zeta_3\\
& \qquad\left.\left.+\left(\frac{1}{2}(k_3-k_4)\underset{(p)}{\cdot}(k_1-k_2) - ik_3\underset{(p,p')}{\cdot}Jk_4\right)\zeta_3\underset{(p')}{\cdot}\zeta_4\right\}\right]\\
& +\left(k_3\leftrightarrow k_4;\zeta_3\leftrightarrow\zeta_4\right)\quad .
\end{aligned}\tag{4.10}$$

Let us give an explicit connection between the g.d.f. and the noncommutative projector soliton, which is a key observation to our work. The g.d.f. is rewritten as

$$\begin{aligned}
D(k_m) \;&=\; \exp\left(-\frac{1}{4}\sum_{I=\frac{p+2}{2}}^{\frac{p'}{2}} \mid \theta^{2I-1,2I}\mid \left(k_{2I-1}k_{2I-1}+k_{2I}k_{2I}\right)\right)\\
&=\; \prod_{I=\frac{p+2}{2}}^{\frac{p'}{2}}\tilde{\phi}_0(k_{2I-1},k_{2I};\theta^{2I-1,2I})\quad .
\end{aligned}\tag{4.11}$$

Observe that

$$\begin{aligned}
2\pi\mid\theta\mid\tilde{\phi}_0 \;&=\; \int d^2x e^{ik_1x^1+ik_2x^2}\phi_0(x^1,x^2;\theta)\quad ,\\
\phi_0(x^1,x^2;\theta) \;&=\; 2e^{-\frac{1}{|\theta|}((x^1)^2+(x^2)^2)}\quad .
\end{aligned}\tag{4.12}$$

Function $\phi_0(x^1,x^2;\theta)$ is the projector soliton solution of the noncommutative scalar field theory discussed in [9]. It satisfies $\phi_0 * \phi_0 = \phi_0$ and is represented as a ground state projector $|0\rangle\langle 0|$ in the Fock space representation of noncommutative algebra $[x^1,x^2]=i\theta$. In [9], ϕ_0 is discussed as a soliton solution of noncommutative scalar field theory in the large θ limit. In our discussion, Fourier transform of ϕ_0 is seen to appear for all values of θ.

Eq. (4.7) tells us that the g.d.f. $D(k_{3m})$ is a form factor of the Dp-brane of size $\sqrt{\mid \beta_I \mid}$ by noncommutative $U(1)$ current, which can be written as

$$\left(\Phi^\dagger \overleftrightarrow{\partial}^{\mu} \Phi, -i\partial_n\left(\Phi^\dagger J^{mn}\Phi\right)\right) \quad , \tag{4.13}$$

using the scalar field $\Phi(x^\mu,x^m)$ discussed in the next section. Putting together this fact and the obsevation of the last paragraph, we identify the Dp-brane in the zero slope limit with the noncommutative soliton. See [10] for this identification from string field theory.

V. Dp brane and the Projector Soliton of Noncommutative Scalar Field Theory

We now give a field theoretic derivation of the properties of the string amplitude in the zero slope limit given by eqs. (4.7), (4.9) and (4.10). We will show that an adequate description is given in perturbation theory of low energy effective action (LEEA) proposed in [2] by specifying proper initial and final states associated with the scalar field $\Phi(x^\mu,x^m)$.

In [2], the following action has been proposed:

$$\begin{aligned}
&S = S_0 + S_1 \ , \\
&\text{with} \quad S_0 = \frac{1}{g_{YM}^2}\int d^{p'+1}x\sqrt{-G}\left\{-\left(D_\mu\Phi\right)^\dagger * \left(D^\mu\Phi\right) - m^2\Phi^\dagger * \Phi - \frac{1}{4}F_{MN} * F^{MN}\right\} \ , \\
&\qquad\quad S_1 = \frac{1}{2g_{YM}^2}\int d^{p'+1}x\sqrt{-G}\Phi^\dagger * F_{mn}J^{mn} * \Phi \ ,
\end{aligned} \tag{5.1}$$

where

$$\begin{aligned}
&D_\mu\Phi = \partial_\mu\Phi - iA_\mu * \Phi \ , \qquad (D_\mu\Phi)^\dagger = \partial_\mu\Phi^\dagger + i\Phi^\dagger * A_\mu \ , \\
&F_{MN} = \partial_M A_N - \partial_N A_M - i\left[A_M,A_N\right]_* \ , \quad \left[A_M,A_N\right]_* = A_M * A_N - A_N * A_M \ .
\end{aligned} \tag{5.2}$$

Here $A_M(x^\mu,x^m)$ is a $(p'+1)$-dimensional vector field which corresponds to noncommutative $U(1)$ photon and $\Phi(x^\mu,x^m)$ is a scalar field which corresponds to the ground state tachyon of the $p-p'$ open string with $m^2 = -\lim_{\alpha'\to 0}(1-\sum_I \nu_I)/2\alpha'$. Reflecting the fact that the tachyon

momenta are constrained to lie in $p+1$ dimensions, the Lorentz index of the kinetic term for the scalar field runs from 0 to p and there is no kinetic term for the remaining $p'-p$ directions. From now on, we set g_{YM} to 1.

It is elementary to compute the three point tree amplitude from $\mathcal{L}_{\rm int}(\Phi, A_M)$:

$$\begin{aligned}
\mathcal{A}_3 &= i\int d^{(p'-p)}K_m^{(f)}\int d^{(p'+1)}x^M\sqrt{-G}\,{}_{\rm sol}\langle\!\langle -K_m^{(f)}|\otimes{}_{\rm tach}\langle -k_{2\mu}| \\
&\quad \Big\{\frac{1}{2}\Phi^\dagger *{}_{\rm vec}\langle 0|F_{mn}J^{mn}|k_{3M}\rangle_{\rm vec}*\Phi \\
&\quad -i\Phi^\dagger\left(*_{\rm vec}\langle 0|A_\mu|k_{3M}\rangle_{\rm vec}*\overrightarrow{\partial}^\mu-\overleftarrow{\partial}^\mu*_{\rm vec}\langle 0|A_\mu|k_{3M}\rangle_{\rm vec}*\right)\Phi\Big\}|k_{1\mu}\rangle_{\rm tach}\otimes|0\rangle\!\rangle_{\rm sol} \\
&= -i\left(\frac{1}{\sqrt{-G}}\right)^2(2\pi)^{p+1}\delta^{(p+1)}\left(\sum_{a=1}^{3}k_{a\mu}\right)\exp\left(\frac{i}{2}\theta^{\mu\nu}k_{1\mu}k_{2\nu}\right)u^*(k_{3m})u(0) \qquad (5.3) \\
&\quad \prod_{a=1,2}\frac{1}{\sqrt{(2\pi)^p2\omega_{\vec{k}_a}}}\frac{1}{\sqrt{(2\pi)^p2|\vec{k}_3|}}\left((k_2-k_1)\underset{(p')}{\cdot}\zeta_3-ik_3\underset{(p,p')}{\cdot}J\zeta_3\right)\quad .
\end{aligned}$$

Here the Fock space associated with the vector and those with the x^μ and x^m dependent part of the scalar field $\Phi(x^\mu, x^m)$ are designated by $vec, tach, sol$ respectively. Eq.(4.7) from string theory and eq.(5.4) computed from eq.(5.1) agree completely provided

$$u^*(k_m)u(0)=D(k_m)\quad ,\quad \text{or}\ \ u(k_m)=D(k_m) \qquad (5.4)$$

The momentum space wave function in soliton sector is identified with the g.d.f. and hence is equal to Fourier image of the distribution of the noncommutative soliton.

Let us next see that the N-point tree amplitude obtained from this field theory contains the g.d.f. whose argument is the total momentum. Carrying out the Wick contractions and using the propagator which contains the delta function, we find that the field theory N point amplitude contains the following factor residing in the soliton sector:

$$\begin{aligned}
&\int d^{(p'-p)}K_m^{(f)}\prod_{a=3}^{L}\left(\int d^{(p'-p)}x_a^m\right){}_{\rm sol}\langle\langle -K_m^{(f)}\,|\,\phi^\dagger(x_3^m)*e^{iq_3\underset{(p,p')}{\cdot}x_3}*\delta^{(p'-p)}(x_3-x_4) \\
&\quad *e^{iq_4\underset{(p,p')}{\cdot}x_4}*\cdots*\delta^{(p'-p)}(x_{L-1}-x_L)*e^{iq_L\underset{(p,p')}{\cdot}x_L}*\phi(x_L^m)|K_m^{(i)}=0\rangle\rangle_{\rm sol}\quad . \qquad (5.5)
\end{aligned}$$

Thanks to the delta function propagator, this equals

$$\begin{aligned}
&= \int d^{(p'-p)}K_m^{(f)}\int d^{(p'-p)}x^m\exp\left(\frac{i}{2}\sum_{\substack{a,b=3\\a<b}}^{L}\sum_{m,n=p+1}^{p'}\theta^{mn}q_{am}q_{bn}\right) \\
&\quad {}_{\rm sol}\langle\langle -K_m^{(f)}|\phi^\dagger(x^m)*\exp\left(i\left(\sum_{a=3}^{L}q_a\right)\underset{(p,p')}{\cdot}x\right)*\phi(x^m)|K_m^{(i)}=0\rangle\rangle_{\rm sol}
\end{aligned}$$

$$= \int d^{(p'-p)} K_m^{(f)} \delta^{(p'-p)} \left(K_m^{(f)} + (\sum_{a=3}^{L} q_a) \right) u^*(-K_m^{(f)}) u(0) \exp \left(\frac{i}{2} \sum_{\substack{a,b=3 \\ a<b}}^{L} \sum_{m,n=p+1}^{p'} \theta^{mn} q_{am} q_{bn} \right)$$

$$= D \left(\sum_{a=3}^{L} k_{am} \right) \exp \left(\frac{i}{2} \sum_{\substack{a,b=3 \\ a<b}}^{L} \sum_{m,n=p+1}^{p'} \theta^{mn} q_{am} q_{bn} \right) \quad . \tag{5.6}$$

Finally, let us check that the tree four point amplitude (the pole part) in fact agrees with string answer. After the Wick contraction and the position space integration, we find

$$\mathcal{A}_4 = (2\pi)^{p+1} \delta^{p+1} \left(\sum_{a=1}^{4} k_{a\mu} \right) \exp \left(\frac{i}{2} \theta^{\mu\nu} k_{1\mu} k_{2\nu} \right) u^*(k_{3M} + k_{4M})\, u(0)$$
$$\left(\frac{1}{\sqrt{-G}} \right)^3 \prod_{a=1,2} \frac{1}{\sqrt{(2\pi)^p 2\omega_{\vec{k}_a}}} \prod_{b=3,4} \frac{1}{\sqrt{(2\pi)^{p'} |\vec{k}_b|}} \left(\mathbf{a}_4^{(t,u)} + \mathbf{a}_4^{(s)} \right) \;, \tag{5.7}$$

where

$$\begin{aligned} \mathbf{a}_4^{(t,u)} &= \frac{-i}{t-m^2} \left\{ (k_2 - (k_1 + k_4)) \underset{(p)}{\cdot} \zeta_3 - i k_3 \underset{(p,p')}{\cdot} J \zeta_3 \right\} \\ &\qquad \left\{ (k_2 + k_3) - k_1) \underset{(p)}{\cdot} \zeta_4 - i k_4 \underset{(p,p')}{\cdot} J \zeta_4 \right\} \exp \left(\frac{i}{2} \theta^{MN} k_{3M} k_{4N} \right) \\ &\quad + (k_3 \leftrightarrow k_4; \zeta_3 \leftrightarrow \zeta_4) \;, \end{aligned} \tag{5.8}$$

$$\begin{aligned} \mathbf{a}_4^{(s)} &= \frac{-i}{s}\, 2 \left[\left((k_2 - k_1) \underset{(p)}{\cdot} \zeta_3 - i(k_3 + k_4) \underset{(p,p')}{\cdot} J \zeta_3 \right) k_3 \underset{(p')}{\cdot} \zeta_4 \right. \\ &\qquad - \left((k_2 - k_1) \underset{(p)}{\cdot} \zeta_4 - i(k_3 + k_4) \underset{(p,p')}{\cdot} J \zeta_4 \right) k_4 \underset{(p')}{\cdot} \zeta_3 \\ &\qquad \left. + \left(\frac{1}{2} (k_3 - k_4) \underset{(p)}{\cdot} (k_1 - k_2) - i k_3 \underset{(p,p')}{\cdot} J k_4 \right) \zeta_3 \underset{(p')}{\cdot} \zeta_4 \right] \exp \left(\frac{i}{2} \theta^{MN} k_{3M} k_{4N} \right) \\ &\quad + (k_3 \leftrightarrow k_4; \zeta_3 \leftrightarrow \zeta_4) \;. \end{aligned} \tag{5.9}$$

This expression agrees with eq.(4.10).

We have thus shown that perturbation theories of two different kinds in fact agree.

References

[1] B. Chen, H. Itoyama, T. Matsuo, and K. Murakami, " $p - p'$ System with B-field, Brane at Angle and Noncommutative Geometry", Nucl. Phys. **B**576(2000)177, hep-th/9910263.

[2] B. Chen, H. Itoyama, T. Matsuo, and K. Murakami, "Worldsheet and Spacetime Properties of $p-p'$ System with B Field and Noncommutative Geometry", Nucl. Phys. B to appear, hep-th/0005283.

[3] B. Chen, H. Itoyama, T. Matsuo, and K. Murakami, " Correspondence between Noncommutative Soliton and Open String/D-brane System via Gaussian Damping Factor", hep-th/0010066

[4] C.G. Callan, C. Lovelace, C.R. Nappi, S.A. Yost, "String Loop Corrections To Beta Functions", Nucl. Phys. **B288** (1987) 525;
A. Abouelsaood, C.G. Callan, C.R. Nappi and S.A. Yost, "Open Strings In Background Gauge Fields", Nucl. Phys. **B280** (1987) 599.
E.S. Fradkin and A.A. Tseytlin, "Nonlinear Electrodynamics From Quantized Strings", Phys. Lett. **163B** (1985) 123.

[5] N. Seiberg and E. Witten, "String Theory and Noncommutative Geometry", JHEP **9812** (1998) 008, hep-th/9908142.

[6] L. Dixon, D. Friedan, E. Martinec and S. Shenker, "The Conformal Field Theory of Orbifolds", Nucl. Phys. **B282** (1987) 13.

[7] E. Gava, K.S. Narain and M.H. Sarmadi, "On the Bound States of p- and $(p+2)$-branes", Nucl. Phys. **B504** (1997) 214, hep-th/9704006.

[8] H. Itoyama and P. Moxhay, "Multiparticle Superstring Tree Amplitudes", Nucl. Phys. **B293** (1987) 685;
Y. Arakane, H. Itoyama, H. Kunitomo and A. Tokura, "Infinity Cancellation, Type I' Compactification and String $\mathcal{S}$-matrix Functional", Nucl. Phys. **B486** (1997) 149, hep-th/9609151.

[9] R. Gopakumar, S. Minwalla and A. Strominger, "Noncommutative Solitons", JHEP **0005** (2000) 020, hep-th/0003160.

[10] J. Harvey, P. Kraus, F. Larsen and E.J. Martinec, "D-branes and Strings as Noncommutaitve Solitons", hep-th/0005031;
J. Harvey, P.Kraus and F. Larsen, "Exact Noncommutative Solitons", hep-th/0010060